ON INVARIANT THEORY OF θ-GROUPS

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INTRODUCTION

This paper is a contribution to Vinberg's theory of θ-groups, or in other words, to Invariant Theory of periodically graded semisimple Lie algebras [Vi1],[Vi2]. One of our main tools is Springer's theory of regular elements of finite reflection groups [Sp], with some recent complements by Lehrer and Springer [LS1], [LS2].

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Throughout, G is a connected and simply connected semisimple algebraic group, $\mathfrak{g}$ is its Lie algebra, and Φ is the Cartan–Killing form on $\mathfrak{g}$; $l = \mathrm{rk}\,\mathfrak{g}$.

$\mathrm{Int}\,\mathfrak{g}$ (resp. $\mathrm{Aut}\,\mathfrak{g}$) is the group of inner (resp. all) automorphisms of $\mathfrak{g}$; $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. For $x \in \mathfrak{g}$, $\mathfrak{z}(x)$ is the centraliser of x in $\mathfrak{g}$.

Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be a periodic grading of $\mathfrak{g}$ and θ the corresponding m^{th} order automorphism of $\mathfrak{g}$. Let G_0 denote the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Invariant Theory of θ-groups deals with orbits and invariants of G_0 acting on $\mathfrak{g}_1$. Its main result is that there is a subspace $\mathfrak{c} \subset \mathfrak{g}_1$ and a finite reflection group $W(\mathfrak{c}, \theta)$ in $\mathfrak{c}$ (the *little Weyl group*) such that $\Bbbk[\mathfrak{g}_1]^{G_0} \simeq \Bbbk[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$. We say that the grading is N-regular (resp. S-regular) if $\mathfrak{g}_1$ contains a regular nilpotent (resp. semisimple) element of $\mathfrak{g}$. The grading is locally free if there is $x \in \mathfrak{g}_1$ such that $\mathfrak{z}(x) \cap \mathfrak{g}_0 = \{0\}$. The same terminology also applies to θ.

In this paper, we obtain some structural results for gradings with these properties and study interrelations of these properties. Section 1 contains some preliminary material on θ-groups and regular elements. In Section 2, we begin with a dimension formula for semisimple G_0-orbits in $\mathfrak{g}_1$. We also prove two "uniqueness" theorems. Recall that $\mathrm{Int}\,\mathfrak{g}$ is the identity component of $\mathrm{Aut}\,\mathfrak{g}$, and it operates on $\mathrm{Aut}\,\mathfrak{g}$ via conjugations. Given $m \in \mathbb{N}$, we prove that each connected component of $\mathrm{Aut}\,\mathfrak{g}$ contains at most one $\mathrm{Int}\,\mathfrak{g}$-orbit consisting of automorphisms of order m that are either N-regular or S-regular and locally free. In Section 3, we show that θ-groups corresponding to N-regular gradings enjoy a

number of good properties. Let $m_1, \ldots, m_l$ be the exponents of $\mathfrak{g}$ and let $\{e, h, f\}$ be a regular $\mathfrak{sl}_2$-triple such that $e \in \mathfrak{g}_1$ and $f \in \mathfrak{g}_{-1}$. For the purposes of this introduction, assume that θ is inner. Set $k_i = \#\{j \mid m_j \equiv i \pmod{m}\}$ $(i \in \mathbb{Z}_m)$, and let ζ be a primitive m^{th} root of unity. We prove that (i) the eigenvalues of θ on $\mathfrak{z}(e)$ are ζ^{m_i} $(1 \leq i \leq l)$, (ii) $\dim \mathfrak{g}_{i+1} - \dim \mathfrak{g}_i = k_{-i-1} - k_i$ for all $i \in \mathbb{Z}_m$, (iii) The restriction homomorphism $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto, (iv) the G_0-action on $\mathfrak{g}_1$ admits a Kostant-Weierstrass (= KW) section. In the general case, the definition of the k_i's becomes more involved, see Eq. (3.3), but the above assertions (ii)–(iv) remains intact.

In Section 4, it is shown that any locally free S-regular grading of $\mathfrak{g}$ is N-regular. This implies that all such gradings admit a KW-section. We also give a formula for dimension of all subspaces $\mathfrak{g}_i$ in the S-regular case. Another result is that $\dim \mathfrak{c} \leq k_{-1}$ for any θ-group. We then show that the G-stable cone $\pi^{-1}\pi(\mathfrak{c}) \subset \mathfrak{g}$ is a normal complete intersection. (Here $\pi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ is the quotient mapping.) In particular, if θ is S-regular or N-regular, then $\overline{G{\cdot}\mathfrak{g}_1}$ is a normal complete intersection. A description of the defining ideal of $\overline{G{\cdot}\mathfrak{g}_1}$ is also given. This normality stuff relies on results of Richardson [Ri]. It is curious to note that in case $m = 2$ (i.e., θ is involutory) S-regularity is equivalent to N-regularity. But for $m > 2$ neither of these properties implies the other.

Section 5 contains a description of the coexponents for little Weyl groups, if θ is both S- and N-regular. This is based on recent results of Lehrer and Springer [LS2].

1. VINBERG'S θ-GROUPS AND SPRINGER'S REGULAR ELEMENTS

Let θ be an automorphism of $\mathfrak{g}$, of finite order m. The automorphism of G induced by θ is also denoted by θ. Let ζ be a fixed primitive m^{th} root of unity. Then θ determines a periodic grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \theta(x) = \zeta^i x\}$. Whenever we want to indicate the dependence of the grading on θ, we shall endow '$\mathfrak{g}_i$' with a suitable superscript. If M is a θ-stable subspace, then $M_i := M \cap \mathfrak{g}_i$. Recall some standard facts on periodic gradings (see [Vi1, §1]):

- $\Phi(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$;
- Φ is non-degenerate on $\mathfrak{g}_i \oplus \mathfrak{g}_{-i}$ $(i \neq 0)$ and on $\mathfrak{g}_0$. In particular, $\mathfrak{g}_0$ is a reductive algebraic Lie algebra and $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$;
- If $x \in \mathfrak{g}_i$ and $x = x_s + x_n$ is its Jordan decomposition, then $x_s, x_n \in \mathfrak{g}_i$.

Let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. The restriction of the adjoint representation of G to G_0 induces a representation ρ_1 of G_0 on $\mathfrak{g}_1$. The linear group

$\rho_1(G_0) \subset GL(\mathfrak{g}_1)$ is called a θ-*group*. The theory of orbits and invariants for θ-groups, which generalizes that for the adjoint representation [Ko] and for the isotropy representation of a symmetric variety [KR], is developed by E.B. Vinberg in [Vi1].

A *Cartan subspace* of $\mathfrak{g}_1$ is a maximal commutative subspace consisting of semisimple elements. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace. Set $N(\mathfrak{c})_0 = \{g \in G_0 \mid \mathrm{Ad}\,(g)\mathfrak{c} = \mathfrak{c}\}$ and $Z(\mathfrak{c})_0 = \{g \in G_0 \mid \mathrm{Ad}\,(g)x = x \text{ for all } x \in \mathfrak{c}\}$. The group $N(\mathfrak{c})_0/Z(\mathfrak{c})_0$ is said to be the *little Weyl group* of the graded Lie algebra, denoted $W(\mathfrak{c}, \theta)$.

The following is a summary of main results in [Vi1].

1.1 Theorem.

    (i)   *All Cartan subspaces in $\mathfrak{g}_1$ are G_0-conjugate;*

    (ii) $W = W(\mathfrak{c}, \theta)$ *is a finite reflection group in* $GL(\mathfrak{c})$;

    (iii)   *Let $x \in \mathfrak{g}_1$. The orbit $G_0 {\cdot} x$ is closed if and only if $G_0 {\cdot} x \cap \mathfrak{c} \neq \varnothing$; the closure $\overline{G_0 {\cdot} x}$ contains the origin if and only if $x \in \mathcal{N}$;*

    (iv)   *The restriction of polynomial functions $\Bbbk[\mathfrak{g}_1] \to \Bbbk[\mathfrak{c}]$ induces an isomorphism $\Bbbk[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \Bbbk[\mathfrak{c}]^W$ (a "Chevalley-type" theorem);*

    (v)   *Each fibre of the quotient mapping $\pi_1 : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0 = \mathrm{Spec}\,\Bbbk[\mathfrak{g}_1]^{G_0}$ consists of finitely many G_0-orbits. The dimension of each fibre is equal to $\dim \mathfrak{g}_1 - \dim \mathfrak{c}$.*

Despite its maturity, the theory of θ-groups still has a vexatious gap. A long-standing conjecture formulated in [Po, n.7], to the effect that any θ-group has an analogue of the section constructed by Kostant for the adjoint representation (see [Ko, n. 4]), is still open. (In [KR], such a section was also constructed for the isotropy representation of a symmetric variety. So that the problem concerns the case $m \geq 3$.) Kostant's section for the adjoint representation is an instance of a more general phenomenon in Invariant Theory, a so-called *Weierstrass section*. The reader is referred to [VP, 8.8] for the general definition of a Weierstrass section and a number of related results. In my opinion, it is more natural to use term Kostant-Weierstrass sections, or *KW-sections* in the context of θ-groups. It was shown in [Pa, Cor. 5] that a KW-section exists whenever $\mathfrak{g}_0$ is semisimple. Below, we will discuss some aspects of KW-sections in a more general situation.

If W be a finite reflection group in a $\Bbbk$-vector space V, then $v \in V$ is called regular if the stabiliser of v in W is trivial. Let σ be an element of finite order in $N_{GL(V)}(W)$. Then σ is called *regular* (in the sense of Springer) if it has a regular eigenvector. The theory of such elements is developed by Springer in [Sp]; for recent results, see [LS1]. Let $f_1, f_2, \ldots, f_l$ be a set of algebraically independent homogeneous generators of $\Bbbk[V]^W$, $l = \dim V$. Set $d_i = \deg f_i$. The f_i's can be chosen so that $\sigma(f_i) = \varepsilon_i f_i$ $(1 \leq i \leq l)$, with suitable roots of unity ε_i. Given a root of unity ζ, we let $V(\sigma, \zeta)$ denote the eigenspace of σ corresponding

to the eigenvalue ζ. The following is a sample of Springer's results, see Theorem 6.4 in [Sp].

1.2 Theorem. *Suppose $V(\sigma, \zeta)$ contains a regular vector. Then*

  (i) $\dim V(\sigma, \zeta) = \#\{j \mid 1 \le j \le l,\ \varepsilon_j \zeta^{d_j} = 1\}$*;*

  (ii) *The centraliser of σ in W, denoted W^σ, is a reflection group in $V(\sigma, \zeta)$, and the restrictions $f_j|_{V(\sigma, \zeta)}$ with $\varepsilon_j \zeta^{d_j} = 1$ form a set of basic invariants for W^σ;*

  (iii) *The eigenvalues of σ in V are $\varepsilon_i^{-1} \zeta^{-d_i+1}$, $i = 1, \ldots, l$;*

  (iv) $\dim V(w\sigma, \zeta) \le \dim V(\sigma, \zeta)$ *for all $w \in W$. If $\dim V(\sigma, \zeta) = \dim V(w\sigma, \zeta)$ for some $w \in W$, then σ and $w\sigma$ are conjugate by an element of W.*

We will apply Springer's theory in the context of θ-groups, when $V = \mathfrak{t}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$, $W = W(\mathfrak{t})$ is the Weyl group of $\mathfrak{t}$, and $\sigma = \theta|_{\mathfrak{t}}$. Obviously, such σ normalises the Weyl group.

<div align="center">2. MISCELLANEOUS RESULTS ON PERIODIC GRADINGS</div>

2.1 Proposition.

  (i) *If $x \in \mathfrak{g}_1$ is semisimple, then $\dim \mathfrak{g}_k - \dim \mathfrak{z}(x)_k$ does not depend on k. In particular, $\dim[\mathfrak{g}_0, x] = \frac{1}{m} \dim[\mathfrak{g}, x]$;*

  (ii) *If $m = 2$, then the relation $\dim[\mathfrak{g}_0, x] = \frac{1}{2} \dim[\mathfrak{g}, x]$ holds for all $x \in \mathfrak{g}_1$.*

Proof. Consider the Kirillov form $\mathcal{K}_x$ on $\mathfrak{g}$. By definition, $\mathcal{K}_x(y, z) = \Phi(x, [y, z])$. From the invariance of Φ one readily deduces that $\operatorname{Ker} \mathcal{K}_x = \mathfrak{z}(x)$. Since $x \in \mathfrak{g}_1$, we have $\mathcal{K}_x(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = -1$. It follows that

$$\dim \mathfrak{g}_k - \dim \mathfrak{z}(x)_k = \dim \mathfrak{g}_{-k-1} - \dim \mathfrak{z}(x)_{-k-1} \text{ for all } k \in \mathbb{Z}_m \ .$$

This already implies (ii). If m is arbitrary, then one obtains a good conclusion only for semisimple elements. Indeed, if $\mathfrak{z}(x)$ is reductive, then $\dim \mathfrak{z}(x)_k = \dim \mathfrak{z}(x)_{-k}$ for all k. Hence $\dim \mathfrak{g}_k - \dim \mathfrak{z}(x)_k$ does not depend on k. $\qquad\square$

Part (ii) is due to Kostant and Rallis, see [KR, Prop. 5].

Definition. A periodic grading (or the corresponding automorphism) of $\mathfrak{g}$ is called S-*regular*, if $\mathfrak{g}_1$ contains a regular semisimple element of $\mathfrak{g}$; N-*regular*, if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. It is called *locally free*, if there exists $x \in \mathfrak{g}_1$ such that $\mathfrak{z}(x)_0 = \{0\}$.

Our aim is to prove two conjugacy theorems for periodic gradings.

2.2 Theorem. *Let θ_1, θ_2 be automorphisms of $\mathfrak{g}$ having the same order and lying in the same connected component of $\operatorname{Aut}\mathfrak{g}$. Suppose the corresponding periodic gradings are* S-*regular and locally-free. Then θ_1, θ_2 are conjugated by means of an element of $\operatorname{Int}\mathfrak{g}$.*

Proof. Let $\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_i^{(1)}$ and and $\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_i^{(2)}$ be the corresponding gradings. It follows from the hypotheses that $\mathfrak{g}_1^{(i)}$ $(i = 1, 2)$ contains a regular semisimple element x_i such that $\mathfrak{z}(x_i)_0 = \{0\}$. Since all Cartan subalgebras are conjugate with respect to Int $\mathfrak{g}$, we may assume that $\mathfrak{z}(x_1) = \mathfrak{z}(x_2) =: \mathfrak{t}$. Set $\sigma_i = \theta_i|_\mathfrak{t} \in \operatorname{Aut}(\mathfrak{t})$. Since $\theta_1\theta_2^{-1}$ is inner, we see that $\sigma_1 = w\sigma_2$ for some $w \in W(\mathfrak{t})$, where $W(\mathfrak{t})$ is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Thus, σ_2 and $w\sigma_2$ are two elements on $GL(\mathfrak{t})$ having a regular eigenvector, with the same eigenvalue. By [Sp, 6.4(iv)], it then follows that σ_1 and σ_2 are $W(\mathfrak{t})$-conjugate. Thus, we may assume that $\theta_1|_\mathfrak{t} = \theta_2|_\mathfrak{t}$. Then $\theta_2\theta_1^{-1} = \operatorname{Ad}(s)$ for some $s \in T = \exp(\mathfrak{t})$. By assumption, $\mathfrak{t}_0 = \{0\}$. Therefore $\theta_2|_T$ has finitely many fixed points. Hence the mapping $(g \in T) \mapsto (\theta_2(g)g^{-1} \in T)$ is onto, and there exists $t \in T$ such that $s = \theta_2(t)t^{-1}$. Then

$$\theta_1 = \operatorname{Ad}(s)^{-1}{\cdot}\theta_2 = \operatorname{Ad}(t){\cdot}\operatorname{Ad}(\theta_2(t^{-1})){\cdot}\theta_2 = \operatorname{Ad}(t){\cdot}\theta_2{\cdot}\operatorname{Ad}(t^{-1}),$$

and we are done. $\qquad\square$

Recall that $x \in \mathcal{N}$ is called *semiregular*, if any semisimple element of the centraliser $Z_G(x)$ belong to the centre of G. The corresponding orbit and $\mathfrak{sl}_2$-triple are also called semiregular. The semiregular $\mathfrak{sl}_2$-triples in simple Lie algebras were classified by E.B. Dynkin in 1952.

2.3 Theorem. *Let θ', θ'' be automorphisms of $\mathfrak{g}$ having the same order and lying in the same connected component of $\operatorname{Aut}\mathfrak{g}$. Suppose there exists a semiregular nilpotent orbit $\mathcal{O} \in \mathfrak{g}$ such that $\mathcal{O} \cap \mathfrak{g}_1' \neq \varnothing$ and $\mathcal{O} \cap \mathfrak{g}_1'' \neq \varnothing$. Then θ', θ'' are conjugated by means of an element of $\operatorname{Int}\mathfrak{g}$.*

Proof. In case $\mathcal{O}$ is the regular nilpotent orbit, a proof is given in [An]. It goes through in our slightly more general setting. For convenience of the reader, we give it here.

Let $e' \in \mathfrak{g}_1' \cap \mathcal{O}$ and $e'' \in \mathfrak{g}_1'' \cap \mathcal{O}$. According to [Vi2, sect. 2], there exist $\mathfrak{sl}_2$-triples $\{e', h', f'\}$, $\{e'', h'', f''\}$ such that $h' \in \mathfrak{g}_0'$, $h'' \in \mathfrak{g}_0''$, $f' \in \mathfrak{g}_{-1}'$, $f'' \in \mathfrak{g}_{-1}''$. By the conjugacy theorem for $\mathfrak{sl}_2$-triples, there exists $\tau \in \operatorname{Int}\mathfrak{g}$ such that $\tau(e') = e''$, $\tau(h') = h''$, $\tau(f') = f''$. Then $\theta'^{-1}\tau^{-1}\theta''\tau$ is inner, and it takes the triple $\{e', h', f'\}$ to itself. Since the centraliser of a semiregular $\mathfrak{sl}_2$-triple in $\operatorname{Int}\mathfrak{g}$ is trivial, we obtain $\theta' = \tau^{-1}\theta''\tau$. $\qquad\square$

3. N-REGULAR PERIODIC GRADINGS

Let $\mathcal{O}^{reg}$ be the regular nilpotent orbit in $\mathfrak{g}$. Recall that a periodic grading (or automorphism) of $\mathfrak{g}$ is N-regular, if $\mathcal{O}^{reg} \cap \mathfrak{g}_1 \neq \varnothing$. Since $\mathcal{O}^{reg}$ is semiregular, Theorem 2.3 says that any connected component of $\operatorname{Aut}\mathfrak{g}$ contains at most one $\operatorname{Int}\mathfrak{g}$-orbit of N-regular automorphisms of a prescribed order. To give a detailed description of the N-regular periodic gradings, some preparatory work is needed.

For any $\gamma \in \Gamma(\mathfrak{g}) := \operatorname{Aut} \mathfrak{g}/\operatorname{Int} \mathfrak{g}$, let C_γ denote the corresponding connected component of $\operatorname{Aut} \mathfrak{g}$. The *index* of (any element of) C_γ is the order of γ in $\Gamma(\mathfrak{g})$. The index of $\mu \in \operatorname{Aut} \mathfrak{g}$ is denoted by $\operatorname{ind} \mu$. Thus, $\operatorname{ord}\gamma = \operatorname{ind} C_\gamma = \operatorname{ind} \mu$ for any $\mu \in C_\gamma$.

Since $\operatorname{Int} \mathfrak{g} \simeq G/\{centre\}$, the group $\Gamma(\mathfrak{g})$ acts on $\Bbbk[\mathfrak{g}]^G$ (or on $\mathfrak{g}/\!\!/G = \operatorname{Spec} \Bbbk[\mathfrak{g}]^G$). Let $\mu \in \operatorname{Aut} \mathfrak{g}$ be arbitrary. Denote by $\overline{\mu}$ the corresponding (finite order) automorphism of $\mathfrak{g}/\!\!/G$.

3.1   Lemma. *The action of $\Gamma(\mathfrak{g})$ on $\mathfrak{g}/\!\!/G$ is effective. In other words, the order of $\overline{\mu}$ equals* $\operatorname{ind} \mu$.

Proof.   It is clear that the order of $\overline{\mu}$ divides $\operatorname{ind} \mu$. To prove the converse, we have to show that if $\overline{\mu}$ is trivial, then μ is inner. Without loss of generality, one may assume that μ is a semisimple automorphism. Then, by a result of Steinberg [St, Thm. 7.5], there is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ such that $\mu(\mathfrak{b}) = \mathfrak{b}$ and $\mu(\mathfrak{t}) = \mathfrak{t}$. Let $W(\mathfrak{t})$ be the Weyl group of $\mathfrak{t}$. Since μ acts trivially on $\Bbbk[\mathfrak{g}]^G \simeq \Bbbk[\mathfrak{t}]^{W(\mathfrak{t})}$, the restriction of μ to $\mathfrak{t}$ is given by an element of $W(\mathfrak{t})$. On the other hand, the relation $\mu(\mathfrak{b}) = \mathfrak{b}$ shows that $\mu|_\mathfrak{t}$ permutes somehow the simple roots corresponding to $\mathfrak{b}$. It follows that μ acts trivially on $\mathfrak{t}$ and therefore μ is inner. $\square$

In the following theorems, we describe N-regular periodic gradings and give some relations for eigenvalues and eigenspaces of θ.

3.2   Theorem (Antonyan). *Fix $m \in \mathbb{N}$, and consider a connected component $C_\gamma \subset \operatorname{Aut} \mathfrak{g}$. Then*

$$\left\{ \begin{array}{l} C_\gamma \text{ contains an N-regular auto-} \\ \text{morphism of order } m \end{array} \right\} \quad \Longleftrightarrow \quad \operatorname{ind} C_\gamma \text{ divides } m.$$

In other words, if a connected component of $\operatorname{Aut} \mathfrak{g}$ contains elements of order m, then it contains an N-regular automorphism of order m.

Proof.   If $\operatorname{ind} C_\gamma$ does not divide m, then C_γ does not contain automorphisms of order m. To prove the converse, we first fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots such that the roots of $\mathfrak{b}$ are positive. For each $\alpha_i \in \Pi$, let e_i be a nonzero root vector. Recall that the finite group $\Gamma(\mathfrak{g})$ is isomorphic to the symmetry group of the Dynkin diagram of $\mathfrak{g}$ [VO, 4.4]. This means that each C_γ contains an automorphism θ_γ such that $\theta_\gamma(\mathfrak{t}) = \mathfrak{t}$ and $\theta_\gamma(e_i) = c_i e_{\bar{\gamma}(i)}$, $i = 1, \ldots, l$, where $\bar{\gamma}$ is a permutation on $\{1, \ldots, l\}$ and $c_i \in \Bbbk \setminus \{0\}$. The permutation $\bar{\gamma}$ represents an automorphism of the Dynkin diagram of $\mathfrak{g}$ and the order of $\bar{\gamma}$ equals $\operatorname{ind} \mu$. Conjugating θ_γ by $\operatorname{Ad}(t)$ for a suitable $t \in T$, we can obtain arbitrary coefficients c_i. Therefore we may assume without loss of generality that $c_1 = \cdots = c_l = \zeta$.

6

Then $\theta = \theta_\gamma$ is N-regular, and of order m. Indeed, $e_1 + \cdots + e_l$ is a regular nilpotent element lying in $\mathfrak{g}_1^{(\theta)}$. Next, θ^m is inner and $\theta^m(e_i) = e_i$ for all i. Hence $\theta^m = id_\mathfrak{g}$. □

Let $F_1, F_2, \ldots, F_l$ be homogeneous algebraically independent generators of $\Bbbk[\mathfrak{g}]^G$, $\deg F_i = d_i$. Set $m_i = d_i - 1$. The numbers $m_1 \ldots, m_l$ are called the *exponents* of $\mathfrak{g}$. Given $\theta \in C_\gamma \subset \mathrm{Aut}\,\mathfrak{g}$, we may choose the F_i's so that $\theta(F_i) = \varepsilon_i F_i$ $(i = 1, \ldots, l)$ for some roots of unity ε_i. We shall say that the ε_i's are the *factors* of θ. Note that the multiset $\{\varepsilon_1, \ldots, \varepsilon_l\}$ depends only on the connected component of $\mathrm{Aut}\,\mathfrak{g}$, containing θ. If $\mathfrak{t}$ is an arbitrary Cartan subalgebra, then $f_i = F_i|_\mathfrak{t}$ $(1 \le i \le l)$ are algebraically independent generators for $\Bbbk[\mathfrak{t}]^{W(\mathfrak{t})}$ and $\sigma(f_i) = \varepsilon_i f_i$, where $\sigma = \theta|_\mathfrak{t}$. So, the ε_i's are also factors in the sense of Springer [Sp, §6]. Given $m \in \mathbb{N}$, we shall exploit two sequences indexed by elements of $\mathbb{Z}_m$. Set

$$(3.3) \quad k_i := \#\{j \mid 1 \le j \le l,\ \zeta^{m_j - i}\varepsilon_j = 1\} \quad \text{and} \quad l_i := \#\{j \mid 1 \le j \le l,\ \zeta^{m_j - i}\varepsilon_j^{-1} = 1\}.$$

In this way, we obtain the numbers satisfying the relation $\sum_{i \in \mathbb{Z}_m} k_i = \sum_{i \in \mathbb{Z}_m} l_i = l$. Given $\theta \in \mathrm{Aut}\,\mathfrak{g}$ $(\theta^m = id_\mathfrak{g})$, an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ is said to be θ-*adapted*, if $\theta(e) = \zeta e$, $\theta(h) = h$, and $\theta(f) = \zeta^{-1} f$. By an extension of the Morozov-Jacobson theorem [Vi2, §2], any nilpotent $e \in \mathfrak{g}_1$ can be included in a θ-adapted $\mathfrak{sl}_2$-triple. Recall that an $\mathfrak{sl}_2$-triple is called *regular*, if $e \in \mathcal{O}^{reg}$.

3.4 Theorem. *Suppose $\theta \in \mathrm{Aut}\,\mathfrak{g}$ is N-regular and of order m. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the factors of θ and $\{e, h, f\}$ a θ-adapted regular $\mathfrak{sl}_2$-triple. Then*

(i) *The eigenvalues of θ on $\mathfrak{z}(e)$ are equal to $\zeta^{m_i}\varepsilon_i^{-1}$, $i = 1, \ldots, l$;*

(ii) *The eigenvalues of θ on $\mathfrak{z}(h)$ (resp. $\mathfrak{z}(f)$) are equal to ε_i^{-1} (resp. $\zeta^{-m_i}\varepsilon_i^{-1}$), $i = 1, \ldots, l$;*

(iii) *$k_i = l_i$ for all $i \in \mathbb{Z}_m$.*

(iv) *$\dim \mathfrak{g}_{i+1} - \dim \mathfrak{g}_i = k_{-i-1} - k_i$.*

(v) *The dimension of a Cartan subspace of $\mathfrak{g}_1$ equals k_{-1}.*

Proof. Set $\mathfrak{a} = \Bbbk e + \Bbbk h + \Bbbk f \simeq \mathfrak{sl}_2$. Let $R(n)$ denote the irreducible $\mathfrak{a}$-module of dimension $n + 1$. It is well known that as $\mathfrak{a}$-module $\mathfrak{g}$ is isomorphic to $\oplus_{i=1}^l R(2m_i)$.

(i) Let ∂_f be the derivation of $\Bbbk[\mathfrak{g}]$ determined by f. Then $q_i = (\partial_f)^{m_i} F_i$ is a linear form on $\mathfrak{g}$. By [Ko, Theorem 6], there exists a basis $x_1, \ldots, x_l$ for $\mathfrak{z}(e)$ such that $[h, x_i] = 2m_i x_i$ and $q_i(x_j) = \delta_{ij}$. Since $\theta(q_i) = \zeta^{-m_i}\varepsilon_i$, the latter implies that $\theta(x_i) = \zeta^{m_i}\varepsilon_i^{-1}$.

(ii) Since x_i is a highest weight vector in $R(2m_i)$, the vectors $h_i := (\mathrm{ad}\,f)^{m_i} x_i$ and $y_i := (\mathrm{ad}\,f)^{2m_i} x_i$ $(i = 1, \ldots, l)$ form a basis for $\mathfrak{z}(h)$ and $\mathfrak{z}(f)$, respectively. Obviously, $\theta(h_i) = \varepsilon_i^{-1}$ and $\theta(y_i) = \zeta^{-m_i}\varepsilon_i^{-1}$.

(Another proof for $\mathfrak{z}(h)$ can be derived from [Sp]. Since h is regular semisimple, $\mathfrak{z}(h)$ is a θ-stable Cartan subalgebra. Set $\sigma = \theta|_{\mathfrak{z}(h)}$. Then σ is a regular element in the sense

of Springer. Indeed, σ normalizes the Weyl group $N_G(\mathfrak{z}(h))/Z_G(\mathfrak{z}(h))$ and h is a regular eigenvector of σ. By [Sp, 6.5(i)], the eigenvalues of σ on $\mathfrak{z}(h)$ are equal to ε_i^{-1}, $i = 1, \ldots, l$.)

(iii) It follows from (i) and (ii) that $\dim \mathfrak{z}(e)_i = l_i$ and $\dim \mathfrak{z}(f)_{-i} = k_i$. It remains to observe that $\dim \mathfrak{z}(e)_i = \dim \mathfrak{z}(f)_{-i}$, since Φ is θ-invariant and yields a nondegenerate pairing between $\mathfrak{z}(e)$ and $\mathfrak{z}(f)$.

(iv) Since x_i is a highest weight vector in $R(m_i)$, it follows from (i) that the eigenvalues of θ in $\mathfrak{g}$ are

$$\{\zeta^i \varepsilon_j^{-1} \mid j = 1, 2, \ldots, l; \ i = m_j, m_j - 1, \ldots, -m_j\} \ .$$

So, the problem of computing the required differences becomes purely combinatorial. Notice that each ε_i is an mth root of unity, so that each eigenvalue is a power of ζ. Let us calculate separately the contribution of each submodule $R(2m_j)$ to the difference $D_i := \dim \mathfrak{g}_{i+1} - \dim \mathfrak{g}_i$. Usually, two consecutive eigenvectors with eigenvalues ζ^{i+1} and ζ^i occur together in $R(2m_j)$; i.e., this has no affect on the difference in question. The exceptions can only occur near the eigenvalues of the highest and the lowest weight vectors in $R(2m_j)$. Namely, if $\zeta^{m_j} \varepsilon_j^{-1} = \zeta^i$, one gains contribution -1 to D_i; if $\zeta^{-m_j} \varepsilon_j^{-1} = \zeta^{i+1}$, one gains contribution $+1$ to D_i. Thus, taking the sum over all irreducible $\mathfrak{a}$-submodules yields

$$D_i = \#\{j \mid \zeta^{-m_j - i - 1} \varepsilon_j^{-1} = 1\} - \#\{j \mid \zeta^{m_j - i} \varepsilon_j^{-1} = 1\} = k_{-i-1} - l_i = k_{-i-1} - k_i \ .$$

(v) By parts (i) and (iv), we have $\dim \mathfrak{z}(e)_0 = \#\{j \mid \zeta^{m_j} \varepsilon_j^{-1} = 1\} = l_0 = k_0$ and $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = k_{-1} - k_0$. Therefore $\dim G_0 {\cdot} e = \dim \mathfrak{g}_0 - k_0$. Since $G {\cdot} e$ is open and dense in $\mathcal{N}$, $G_0 {\cdot} e$ is a nilpotent orbit in $\mathfrak{g}_1$ of maximal dimension. By Theorem 1.1(v), $\dim G_0 {\cdot} e$ is also maximal among dimensions of *all* G_0-orbits in $\mathfrak{g}_1$. Thus,

$$\dim \mathfrak{c} = \dim \mathfrak{g}_1 /\!/ G_0 = \dim \mathfrak{g}_1 - \dim G_0 {\cdot} e = k_{-1} \ .$$

$\square$

In the next claim, we regard $\{0, 1, \ldots, m - 1\}$ as a set of representatives for $\mathbb{Z}_m$.

3.5 Corollary. $\dim \mathfrak{g}_0 = \dfrac{1}{m}\Big(\dim \mathfrak{g} + \sum_{i=0}^{m-1} (m - 1 - 2i)k_i\Big).$

Proof.   Write the relations of Theorem 3.4(iv) in the form $\dim \mathfrak{g}_{i+1} - \dim \mathfrak{g}_i = k_{m-i-1} - k_i$, $0 \le i \le m - 1$. Together with the equality $\sum_{i=0}^{m-1} \dim \mathfrak{g}_i = \dim \mathfrak{g}$, these form a system of m linear equations with m indeterminates $\{\dim \mathfrak{g}_i\}$. $\square$

Utility of N-regularity is explained by the fact that this allows us describe the algebra of invariants $\Bbbk[\mathfrak{g}_1]^{G_0}$ and guarantee the existence of a KW-section. Let us briefly recall the last subject. An affine subspace $\mathcal{A} \subset \mathfrak{g}_1$ is called a KW-section if the restriction of $\pi_1 : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!/ G_0$ to $\mathcal{A}$ is an isomorphism. By Theorem 1.1(iii), such an $\mathcal{A}$ contains a unique

nilpotent element. So that $\mathcal{A}$ is of the form $v + L$, where $\{v\} = \mathcal{A} \cap \mathcal{N}$ and L is a linear subspace of $\mathfrak{g}_1$.

3.6 Theorem. *Suppose θ is N-regular and of order m. Let $\{e, h, f\}$ a θ-adapted regular $\mathfrak{sl}_2$-triple. Then*

(i) *The restriction homomorphism $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto. Moreover, $\Bbbk[\mathfrak{g}_1]^{G_0}$ is freely generated by the restriction to $\mathfrak{g}_1$ of all basic invariants F_j such that $\zeta^{m_j+1}\varepsilon_j = 1$.*

(ii) *$e + \mathfrak{z}(f)_1$ is a KW-section in $\mathfrak{g}_1$.*

Proof.   (i) Choose a numbering of basic invariants so that the relation $\zeta^{m_i+1}\varepsilon_i = 1$ holds precisely for $i \le a$. Observe that $a = k_{-1}$. It is immediate that F_i vanishes on $\mathfrak{g}_1$ unless $i \le k_{-1}$. For,

$$\varepsilon_i F_i(x) = (\theta(F_i))(x) = F_i(\theta^{-1}(x)) = F_i(\zeta^{-1}x) = \zeta^{-m_i-1}F_i(x) \text{ for all } x \in \mathfrak{g}_1 .$$

(Recall that $d_i = m_i + 1$.) Our aim is to show that $\bar{F}_i := F_i|_{\mathfrak{g}_1}$ $(1 \le i \le k_{-1})$ generate $\Bbbk[\mathfrak{g}_1]^{G_0}$. (By Theorem 1.1, $\Bbbk[\mathfrak{g}_1]^{G_0}$ is a polynomial algebra in $\dim \mathfrak{c}$ variables, i.e., in our case in k_{-1} variables.) A standard fact of $\mathfrak{sl}_2$-theory says that $\mathfrak{z}(f) \oplus [\mathfrak{g}, e] = \mathfrak{g}$. By a famous result of Kostant [Ko], $(dF_i)_e$ are linearly independent as elements of $\mathfrak{g}^*$ and their images in $\mathfrak{z}(f)^*$ form a basis for $\mathfrak{z}(f)^*$. Therefore, restricting the differentials of basic invariants to $\mathfrak{z}(f)_1$, one obtains a basis for $\mathfrak{z}(f)_1^*$. The preceding exposition shows $(d\bar{F}_i)_e = (dF_i)_e|_{\mathfrak{g}_1} = 0$ unless $1 \le i \le k_{-1}$. On the other hand, it follows from Theorem 3.4(i) that $\dim \mathfrak{z}(f)_1 = k_{-1}$. Hence $(d\bar{F}_i)_e$ $(1 \le i \le k_{-1})$ are linearly independent and $\bar{F}_i$ are algebraically independent. Furthermore, the linear independence of differentials implies that each $\bar{F}_i$ is a member of minimal generating system for $\Bbbk[\mathfrak{g}_1]^{G_0}$, since e lies in the zero locus of all homogeneous G_0-invariants of positive degree. This completes the proof.

(ii) It is a standard consequence of the fact that $\{\bar{F}_i\}$ generate $\Bbbk[\mathfrak{g}_1]^{G_0}$ and $(d\bar{F}_i)_e$ $(1 \le i \le k_{-1})$ are linearly independent, see e.g. [Pa, § 3]. $\square$

Remark. If θ is inner, then $\varepsilon_i = 1$ for all i, and the previous exposition simplifies considerably. In this case, we also have $k_i = \#\{j \mid m_j \equiv i \pmod{m}\}$.

3.7 Corollary. *Suppose θ is inner and N-regular. Then $\Bbbk[\mathfrak{g}_1]^{G_0}$ is freely generated by those $F_i|_{\mathfrak{g}_1}$ whose degree is divisible my m.*

4. APPLICATIONS AND EXAMPLES

In this section, we demonstrate some applications of Springer's theory of regular elements to θ-groups. Maintain the notation of the previous section. In particular, to any $\theta \in \operatorname{Aut} \mathfrak{g}$, of order m, we associate the factors ε_i $(i = 1, \ldots, l)$, which depend only on

the connected component of $\mathrm{Aut}\,\mathfrak{g}$ that contains θ, and then the numbers $k_i = k_i(\theta, m)$ $(i \in \mathbb{Z}_m)$, which are defined by Eq. (3.3).

4.1 Lemma. *Let $\theta \in \mathrm{Aut}\,\mathfrak{g}$ be any automorphism of order m. Then $\dim \mathfrak{g}_1 /\!\!/ G_0 \le k_{-1}$.*

Proof. Take a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$. Let $\mathfrak{t}$ be any θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{c}$. Because of the maximality of $\mathfrak{c}$, we have $\mathfrak{t}_1 = \mathfrak{c}$. Set $\sigma = \theta|_{\mathfrak{t}}$. Then $\sigma \in N_{GL(\mathfrak{t})}(W(\mathfrak{t}))$ and $\mathfrak{c} = \mathfrak{t}(\sigma, \zeta)$. By [Sp, 6.2(i)], one has $\dim \mathfrak{c} \le k_{-1}$ (our k_{-1} is $a(d, \sigma)$ in Springer's paper). $\qquad\qquad\square$

It is shown in the previous section that the k_i's play a significant rôle in the context of N-regular gradings. Now we show that these numbers also relevant to S-regular gradings.

4.2 Theorem. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be an S-regular grading and θ the corresponding automorphism. Then*

(i) *$k_i = k_{-i}$ and $\dim \mathfrak{g}_1 /\!\!/ G_0 = k_{-1}$.*
(ii) *$\dim \mathfrak{g}_0 - k_0 = (\dim \mathfrak{g} - l)/m$, and $\dim \mathfrak{g}_i + k_0 = \dim \mathfrak{g}_0 + k_i$.*
(iii) *If θ is also locally free, then $\mathcal{O}^{reg} \cap \mathfrak{g}_1 \ne \varnothing$, i.e., θ is N-regular.*

Proof. 1. Let $x \in \mathfrak{g}_1$ be a regular semisimple element. Set $\mathfrak{t} = \mathfrak{z}(x)$. It is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Set $\sigma := \theta|_{\mathfrak{t}}$. Because σ originates from an automorphism of $\mathfrak{g}$, it normalizes $W(\mathfrak{t})$, the Weyl group of $\mathfrak{t}$. Furthermore x is a regular eigenvector of σ whose eigenvalue is ζ. Thus, σ is a regular element of $GL(\mathfrak{t})$ in the sense of Springer, and we conclude from [Sp, 6.4(v)] that the eigenvalues of σ are equal to $\zeta^{-m_i} \varepsilon_i^{-1}$ $(i = 1, \ldots, l)$. It follows that $\dim \mathfrak{t}_i = k_{-i}$. Since $\Phi|_{\mathfrak{t}}$ is nondegenerate and σ-stable, $k_{-i} = k_i$. It is also clear that $\mathfrak{t}_1$ is a Cartan subspace of $\mathfrak{g}_1$.

2. These two relations follow from Proposition 2.1(i) applied to x.

3. As the grading is locally-free, $k_0 = \dim \mathfrak{z}(x)_0 = 0$. Let $\tilde{\theta}$ be an N-regular automorphism of order m that lies in the *same* connected component of $\mathrm{Aut}\,\mathfrak{g}$ as θ (cf. Theorem 3.2). Let $\mathfrak{g} = \oplus_i \tilde{\mathfrak{g}}_i$ be the corresponding grading. Let $\mathfrak{c} \subset \tilde{\mathfrak{g}}_1$ be a Cartan subspace. By Theorem 3.4(v), $\dim \mathfrak{c} = k_{-1}$. Let $\tilde{\mathfrak{t}}$ be any $\tilde{\theta}$-stable Cartan subalgebra containing $\mathfrak{c}$. Then, by the definition of a Cartan subspace, we have $\mathfrak{c} = \tilde{\mathfrak{t}}_1$. Conjugating $\tilde{\theta}$ by a suitable inner automorphism, we may assume that $\mathfrak{t} = \tilde{\mathfrak{t}}$. Set $\tilde{\sigma} = \tilde{\theta}|_{\mathfrak{t}}$. Since $\theta \tilde{\theta}^{-1}$ is inner by the construction, $\sigma \tilde{\sigma}^{-1} \in W(\mathfrak{t})$. Thus, we have the following:

σ *has finite order, $\sigma W(\mathfrak{t}) \sigma^{-1} = W(\mathfrak{t})$, $\sigma = w\tilde{\sigma}$ for some $w \in W(\mathfrak{t})$, and $\dim \mathfrak{t}_1 = \dim \tilde{\mathfrak{t}}_1$.*

Since $\mathfrak{t}_1$ contains a regular vector, Theorem 6.4(ii),(iv) from [Sp] applies. It asserts that σ and $\tilde{\sigma}$ are conjugate by an element of $W(\mathfrak{t})$. It follows that $\dim \mathfrak{t}_i = \dim \tilde{\mathfrak{t}}_i$ for all i, and $\tilde{\mathfrak{t}}_1$ contains a regular vector, too. Thus, $\tilde{\theta}$ is S-regular and locally free as well. Finally,

applying Theorem 2.2 to θ and $\tilde{\theta}$, we conclude that these two are conjugate by an element of $\mathrm{Int}\,\mathfrak{g}$. Hence θ is also N-regular. $\qquad\square$

*Remark.* If θ is not assumed to be locally free, then part (iii) can be false, see example below.

Combining Theorem 3.6(ii) and Theorem 4.2(iii), we obtain

4.3 Corollary. *If θ is S-regular and locally free, then the corresponding θ-group admits a KW-section.*

Examples show that $\mathcal{N} \cap \mathfrak{g}_1$, the null-fibre of π_1, is often reducible. Any KW-section, if it exists, must meet one of the irreducible components of $\mathcal{N} \cap \mathfrak{g}_1$. It turns out, however, that some components are 'good' and some are 'bad' in this sense. It may happen that there is only one irreducible component that can be used for constructing a KW-section. It is worth noting in this regard that, in case θ is involutory, all irreducible components of $\mathcal{N} \cap \mathfrak{g}_1$ are 'good', see [KR, Theorem 6].

4.4 Example. Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathbb{E}_6$. Consider two inner automorphisms θ_1, θ_2 of $\mathfrak{g}$ that are defined by the following Kac's diagrams:

$$\theta_1: \qquad\qquad\qquad \theta_2:$$
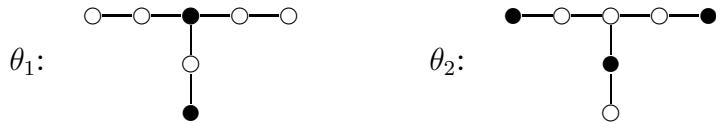
The reader is referred to [VO, 4.7] or [Vi1, § 8] for a thorough treatment of Kac's diagrams of periodic automorphisms. Here we give only partial explanations:

- (The conjugacy class of) a periodic *inner* automorphism of a simple Lie algebra $\mathfrak{g}$ is represented by the corresponding *affine* Dynkin diagram, with white and black nodes.
- The semisimple part of $\mathfrak{g}_0$ is given by the subdiagram consisting of white nodes.
- Dimension of the centre of $\mathfrak{g}_0$ equals the number of black nodes minus 1.
- The order of θ is equal to the sum of those coefficients of the affine Dynkin diagram that correspond to the black nodes.
- Each black node represents an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$, so that the number of black nodes is equal to the number of irreducible summands of $\mathfrak{g}_1$. (We do not give here a general recipe for describing the $\mathfrak{g}_0$-module $\mathfrak{g}_1$.)

It follows that both automorphisms under consideration have order 4, $G_0^{(1)} = A_2 \times A_2 \times A_1 \times \mathbb{k}^*$, and $G_0^{(2)} = A_3 \times A_1 \times (\mathbb{k}^*)^2$.

The $G_0^{(1)}$-module $\mathfrak{g}_1^{(1)}$ has 2 summands: tensor product of simplest representations of all simple factors (dimension 18) plus 2-dimensional representation of A_1. The weights of

$\Bbbk^*$ on these summands, say μ_1 and μ_2, satisfy the relation $3\mu_1 + \mu_2 = 0$ (in the additive notation).

The $G_0^{(2)}$-module $\mathfrak{g}_1^{(2)}$ has 3 summands: simplest representation of A_3 plus its dual plus tensor product of the second fundamental representation of A_3 and the simplest representation of A_1. The weights of $(\Bbbk^*)^2$ on these summands, say μ_i ($i = 1, 2, 3$), satisfy the relation $\mu_1 + \mu_2 + 2\mu_3 = 0$ (in the additive notation).

We have $\dim \mathfrak{g}_0^{(i)} = \dim \mathfrak{g}_1^{(i)} = 20$, $i = 1, 2$. A direct computation shows in both cases that the action $G_0^{(i)} : \mathfrak{g}_1^{(i)}$ is stable, and stabiliser in general position is a 2-dimensional torus. Hence $\dim G_0^{(i)} \cdot x_i = 18$ for a generic (semisimple) $x_i \in \mathfrak{g}_1^{(i)}$ and therefore $\dim G \cdot x_i = 72$ by Proposition 2.1. Notice that $72 = \dim \mathfrak{g} - \mathrm{rk}\,\mathfrak{g}$. Thus, both θ_1 and θ_2 are S-regular but not locally free. Clearly, these are not conjugate. This proves that the assumption of being locally free cannot be dropped in Theorem 2.2.

It is not hard to compute directly that the degrees of basic $G_0^{(i)}$-invariants are equal to $8, 12$ for θ_1 and $4, 8$ for θ_2. As the degrees of $\mathbb{E}_6$ are $2, 5, 6, 8, 9$ and 12, we see that the restriction $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1^{(2)}]^{G_0^{(2)}}$ is not onto. Hence θ_2 is not N-regular. This proves that the assumption of being locally free cannot be dropped in Theorem 4.2(ii). By the way, θ_1 is N-regular.

Let θ be an arbitrary periodic automorphism and let $G_0 : \mathfrak{g}_1$ be the corresponding θ-group, with a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$ and the little Weyl group $W(\mathfrak{c}, \theta)$. The isomorphism 1.1(iv) means that $G_0 \cdot x \cap \mathfrak{c} = W(\mathfrak{c}, \theta) \cdot x$ for all $x \in \mathfrak{c}$. It is not however always true that $G \cdot x \cap \mathfrak{c} = G_0 \cdot x \cap \mathfrak{c}$ for all $x \in \mathfrak{c}$. A similar phenomenon can be seen on the level of Weyl groups, as follows. Let $\mathfrak{t}$ be a θ-stable Cartan subalgebra such that $\mathfrak{t}_1 = \mathfrak{c}$. Write W for the Weyl group $N_G(\mathfrak{t})/Z_G(\mathfrak{t})$. Set $W_1 = N_W(\mathfrak{c})/Z_W(\mathfrak{c})$. It is easily seen that $W(\mathfrak{c}, \theta)$ is isomorphic to a subgroup of W_1 (as all Cartan subalgebras of $\mathfrak{z}_\mathfrak{g}(\mathfrak{c})$ are $Z_G(\mathfrak{c})$-conjugate), but these two groups can be different in general. We give below a sufficient condition for the equality to hold.

Let $\pi : \mathfrak{g} \to \mathfrak{g}/\!/G$ be the quotient mapping. By [Ko], it is known that, for $\xi \in \mathfrak{g}/\!/G$, the fibre $\pi^{-1}(\xi)$ is an irreducible normal complete intersection in $\mathfrak{g}$ of codimension l. The complement of the dense G-orbit in $\pi^{-1}(\xi)$ is of codimension at least 2. Using a result of Richardson [Ri] and an extension of Springer's theory to non-regular elements [LS1], we prove normality of some G-stable cones in $\mathfrak{g}$ associated with θ-groups.

4.5 Theorem. *Suppose θ satisfies the relation $\dim \mathfrak{g}_1/\!/G_0 = k_{-1}$. Then, for any Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$, we have*

$$\pi^{-1}(\pi(\mathfrak{c})) = \bigcap_{i:\ \varepsilon_i \zeta^{d_i} \neq 1} \{x \in \mathfrak{g} \mid F_i(x) = 0\}.$$

This variety is irreducible, normal, and Cohen-Macaulay. Furthermore, its ideal in $\Bbbk[\mathfrak{g}]$ is generated by the above basic invariants F_i, i.e., $\pi^{-1}(\pi(\mathfrak{c}))$ is a complete intersection.

Proof. Let $\mathfrak{t}$ be a θ-stable Cartan subalgebra containing $\mathfrak{c}$ and W the corresponding Weyl group. Set $\sigma = \theta|_{\mathfrak{t}}$. By [LS1, 5.1], W_1 (which is not necessarily the same as either W^σ or $W(\mathfrak{c},\theta)$) is a reflection group in $\mathfrak{c}$ and the functions $F_i|_{\mathfrak{c}}$ with $\varepsilon_i\zeta^{d_i} = 1$ form a set of basic invariants for W_1 (our k_{-1} is $a(d,\sigma)$ in [LS1]). In other words, $\Bbbk[\mathfrak{c}]^{W_1}$ is a graded polynomial algebra and the restriction mapping $\Bbbk[\mathfrak{t}]^W \to \Bbbk[\mathfrak{c}]^{W_1}$ is onto. This means that Theorem B in [Ri, §5] applies here, and we may conclude that $X := \pi^{-1}(\pi(\mathfrak{c}))$ is normal and Cohen-Macaulay. Furthermore, Lemma 5.3 in loc. cit. says that X is irreducible; and the argument in p. 250 in loc. cit shows that the ideal of X is generated by the required basic invariants F_i. □

We have proved before that the hypothesis of Theorem 4.5 is satisfied for S-regular or N-regular gradings. However, in these cases some more precise information is available.

4.6   Theorem. *Suppose $\theta \in \operatorname{Aut}\mathfrak{g}$ is N-regular. Then*

(i) $G{\cdot}x \cap \mathfrak{g}_1 = G_0{\cdot}x$ *for all $x \in \mathfrak{c}$. In particular, $G{\cdot}x \cap \mathfrak{c} = G_0{\cdot}x \cap \mathfrak{c}$;*

(ii) $W(\mathfrak{c},\theta) = W_1$;

(iii) $\pi^{-1}(\pi(\mathfrak{c})) = \overline{G{\cdot}\mathfrak{g}_1}$ *and all assertions of Theorem 4.5 hold for this variety.*

Proof. There are two isomorphisms given by restriction

$$\Bbbk[\mathfrak{g}]^G \xrightarrow{\ \sim\ } \Bbbk[\mathfrak{t}]^W \text{ (Chevalley) and } \Bbbk[\mathfrak{g}_1]^{G_0} \xrightarrow{\ \sim\ } \Bbbk[\mathfrak{c}]^{W(\mathfrak{c},\theta)} \text{ (Vinberg, see 1.1).}$$

Since θ is N-regular, $res_{\mathfrak{g},\mathfrak{g}_1} : \Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto by Theorem 3.6(i). It follows that the restriction mapping $res_{\mathfrak{t},\mathfrak{c}} : \Bbbk[\mathfrak{t}]^W \to \Bbbk[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$ is onto, too. In the geometric form, the ontoness of $res_{\mathfrak{g},\mathfrak{g}_1}$ yields the closed embedding $\mathfrak{g}_1/\!\!/G_0 \hookrightarrow \mathfrak{g}/\!\!/G$. Because the points of such (categorical) quotients parametrise the closed orbits and the closed G_0-orbits in $\mathfrak{g}_1$ are those meeting $\mathfrak{c}$, the above embedding is equivalent to the fact that $G{\cdot}x \cap \mathfrak{g}_1 = G_0{\cdot}x$ for all $x \in \mathfrak{c}$. This gives (i). Similarly, the ontoness of $res_{\mathfrak{t},\mathfrak{c}}$ yields the equality $W(\mathfrak{c},\theta){\cdot}x = W{\cdot}x \cap \mathfrak{c}$ for all $x \in \mathfrak{c}$. Since $W(\mathfrak{c},\theta){\cdot}x \subset W_1{\cdot}x \subset W{\cdot}x \cap \mathfrak{c}$, part (ii) follows.

Let us prove (iii). Set $X := \pi^{-1}(\pi(\mathfrak{c}))$. It is a closed G-stable cone in $\mathfrak{g}$. By [Ri, 5.3], X is irreducible. Since $\overline{G{\cdot}\mathfrak{g}_1}$ is irreducible and $\overline{G{\cdot}\mathfrak{g}_1} \subset X$, it suffices to verify that $\dim X = \dim \overline{G{\cdot}\mathfrak{g}_1}$. Because each fibre of π is of dimension $\dim\mathfrak{g} - l$ and $\dim\pi(\mathfrak{c}) = \dim\mathfrak{c}$, we obtain $\dim X = \dim\mathfrak{c} + \dim\mathfrak{g} - l$. On the other hand, regular elements are dense in $\mathfrak{g}_1$, since θ is N-regular. Therefore $\overline{G{\cdot}\mathfrak{g}_1}$ contains a $\dim\mathfrak{c}$–parameter family of G-orbits of dimension $\dim\mathfrak{g} - l$. This yields the required equality. In view of 3.4(v), Theorem 4.5 applies here. □

Remark. For the S-regular locally free gradings, the coincidence of $W(\mathfrak{c},\theta)$ and W_1 was proved in [Vi1, Prop. 19]. (In that case $W_1 = W^\sigma$ for $\sigma = \theta|_{\mathfrak{z}(\mathfrak{c})}$.) Therefore, in view of Theorem 4.2(iii), the equality 4.6(ii) is an extension of that result of Vinberg.

4.7 Proposition. *Suppose θ is S-regular. Let $\mathfrak{c}$ be any Cartan subspace in $\mathfrak{g}_1$. Then*

$$\pi^{-1}(\pi(\mathfrak{c})) = \overline{G\cdot\mathfrak{g}_1} = \overline{G\cdot\mathfrak{c}}$$

and all assertions of Theorem 4.5 hold.

Proof.   The first equality stems from the presence of regular elements in $\mathfrak{g}_1$ (cf. the proof of 4.6(iii)). Since regular semisimple elements are dense in $\mathfrak{g}_1$, $\overline{G_0\cdot\mathfrak{c}} = \mathfrak{g}_1$. This gives the second equality. In view of 4.2(i), Theorem 4.5 applies here.   □

4.8 Examples. 1. Consider again the automorphism θ_2 from Example 4.4. As we already know, θ_2 is not N-regular and the ontoness of $res_{\mathfrak{g},\mathfrak{g}_1}$ fails here. The latter shows that 4.6(i) does not hold. Since $\mathfrak{c}$ contains regular elements, $Z_W(\mathfrak{c}) = \{1\}$. It is easily seen that W_1 is isomorphic to W^σ, the centraliser of $\sigma = \theta|_{\mathfrak{t}}$ in W. Springer's theory [Sp, § 4] says that $W^\sigma \subset GL(\mathfrak{c})$ is a finite reflection group whose degrees are those degrees of W that are divisible by m, i.e., $8, 12$. Hence $\#W_1 = 96$ and $\#W(\mathfrak{c}, \theta) = 48$. Thus, 4.6(ii) fails, too. However, $\pi^{-1}(\pi(\mathfrak{c}))$ is normal, Cohen-Macaulay, etc., and the reason is that θ_2 is S-regular.

2. It really may happen that θ is neither N-regular nor S-regular, but the equality $\dim \mathfrak{g}_1/\!/G_0 = k_{-1}$ holds. Let $\mathfrak{g}$ be a simple Lie algebra of type $\mathbb{E}_7$. Consider the inner automorphism θ that is determined by the following Kac's diagram:

θ:
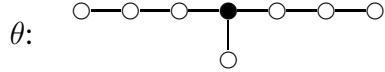
Then the order of θ is 4, $G_0 = A_3 \times A_3 \times A_1$, and $\dim \mathfrak{g}_1 = 32$. Here $k_{-1} = \dim \mathfrak{g}_1/\!/G_0 = 2$, so that Theorem 4.5 applies. On the other hand, $(\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g})/4 \notin \mathbb{N}$. Hence θ is not S-regular. It can be shown that the maximal nilpotent orbit meeting $\mathfrak{g}_1$ is of dimension 120 (its Dynkin-Bala-Carter label is E_6), i.e., θ is not N-regular.

5. EXPONENTS AND COEXPONENTS OF LITTLE WEYL GROUPS

We keep the previous notation. Here we briefly discuss some other consequences of [LS1], [LS2] for θ-groups.

Recall the definition of (co)exponents. If $\tilde{W}$ is a reflection group in $\mathfrak{c}$, then $(\Bbbk[\mathfrak{c}] \otimes N)^{\tilde{W}}$ is a graded free $\Bbbk[\mathfrak{c}]^{\tilde{W}}$-module for any $\tilde{W}$-module N. The *exponents* (resp. *coexponents*) of $\tilde{W}$ are the degrees of a set of free homogeneous generators for this module, if $N = \mathfrak{c}^*$ (resp. $N = \mathfrak{c}$). As is well known, if $\{\tilde{d}_i\}$ are the degrees of basic invariants in $\Bbbk[\mathfrak{c}]^{\tilde{W}}$, then $\{\tilde{d}_i - 1\}$ are the exponents.

The theory of Lehrer and Springer gives a description of coexponents for the subquotient $W_1 = N_W(\mathfrak{c})/Z_W(\mathfrak{c})$ under the constraint $\dim \mathfrak{c} = k_{-1}$, while the theory of θ-groups deals with the group $W(\mathfrak{c}, \theta) = N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$. By 3.4(v) and 4.6(ii), we know that $\dim \mathfrak{c} = k_{-1}$ and $W_1 = W(\mathfrak{c}, \theta)$ whenever θ is N-regular. Thus, N-regularity allows us to exploit the theory of Lehrer and Springer in the context of θ-groups. However, to use that theory in full strength, we need the constraint that θ is S-regular, too.

5.1 Proposition. *Suppose $\theta \in \mathrm{Aut}\,\mathfrak{g}$ is N-regular and let $\{e, h, f\}$ be a θ-adapted regular $\mathfrak{sl}_2$-triple.*

  (i) *The exponents of $W(\mathfrak{c}, \theta)$ correspond to the eigenvalues of θ on $\mathfrak{z}(e)_{-1}$. More precisely, m_j is an exponent of $W(\mathfrak{c}, \theta)$ if and only if $\varepsilon_j \zeta^{m_j} = \zeta^{-1}$;*
  (ii) *If θ is also S-regular, then the coexponents of $W(\mathfrak{c}, \theta)$ correspond to the eigenvalues of θ on $\mathfrak{z}(e)_1$. More precisely, m_j is a coexponent of $W(\mathfrak{c}, \theta)$ if and only if $\varepsilon_j \zeta^{m_j} = \zeta$.*

Proof. (i) This part is essentially contained in Theorem 3.6(i).
(ii) A description of coexponents for subfactors of the form $N_W(\mathfrak{c})/Z_W(\mathfrak{c})$, if $\mathfrak{c}$ contains a regular vector, is due to Lehrer and Springer. However, an explicit formulation was given only in the "untwisted" case (see Theorem C in [LS2]), which in our setting correspond to the case where θ is inner. A general statement can be extracted from the discussion in [LS2, §4], especially from Propositions 4.6, 4.8, and Theorem D.

The connection with θ-eigenvalues on $\mathfrak{z}(e)$ follows from Theorem 3.4. □

Remark. If θ is N-regular, then formulas of Section 3 shows that $k_{-1} = \dim \mathfrak{c} = \dim \mathfrak{z}(e)_{-1}$ and $k_1 = \dim \mathfrak{z}(e)_1$. S-regularity guarantee us that $k_{-1} = k_1$ and, even stronger, that $k_{-i} = k_i$ for all i (see (4.2)). However it can happen that $k_1 > k_{-1}$. For instance, if θ is an N-regular inner automorphism of $\mathbb{E}_6$, of order 5, then $k_1 = 2$ and $k_{-1} = 1$. In this case, it is not clear how to characterise the coexponents for $W(\mathfrak{c}, \theta)$. Another difficulty may occur if θ is S-regular, but not N-regular. Here $k_{-1} = k_1$, but the groups W_1 and $W(\mathfrak{c}, \theta)$ can be different.

5.2 Example. The exceptional Lie algebra $\mathfrak{g} = \mathbb{E}_6$ has an outer automorphism θ of order 4 such that $G_0 = A_3 \times A_1$. This automorphism is determined by the following Kac's diagram

(the underlying graph is the Dynkin diagram of type $E_6^{(2)}$) :   ○——○——○⇐●——○ .
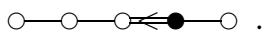
Here $\mathfrak{g}_1$ is the tensor product of a 10-dimensional representation of A_3 (with highest weight $2\varphi_1$) and the simplest representation of A_1. We have $\dim \mathfrak{g}_0 = 18$ and $\dim \mathfrak{g}_1 = 20$. It is easily seen that the G_0-representation on $\mathfrak{g}_1$ is locally free, hence $\dim \mathfrak{g}_1 /\!\!/ G_0 = \dim \mathfrak{c} = 2$. Using Proposition 2.1(i), we conclude that θ is S-regular and then, by Theorem 4.2(iii), that

θ is also N-regular. In order to use the preceding Proposition, one has to know the factors $\{\varepsilon_i\}$. In this case the pairs (m_i, ε_i) $(1 \leq i \leq 6)$ are $(1, 1), (4, -1), (5, 1), (7, 1), (8, -1), (11, 1)$. Then an easy calculation shows that the exponents of $W(\mathfrak{c}, \theta)$ are $7, 11$ and the coexponents are $1, 5$. Then looking through the list of the irreducible finite reflection groups, one finds that here $W(\mathfrak{c}, \theta)$ is Group 8 in the Shephard-Todd numbering. (A list including both the exponents and the coexponents is found in [OS, Table 2]).

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